

On the gradient flows on Finsler manifolds

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Abstract

The purpose of this article is to provide a general overview of curvature functional in Finsler geometry and use its information to introduce the gradient flow on Finsler manifolds. For this purpose, we first prove that the space of Finslerian metrics is a Riemannian manifold. Then it is given a decomposition for the tangent space of this Riemannian manifold by means of Riemannian metric and the Berger-Ebin theorem. Next, Finslerian functional is introduced and show that Akbar-Zadeh curvature functional is the example of Finslerian functional. After that, the critical points of Akbar-Zadeh functional are found in two situations. Based on the constant Indicatrix volume and restricting its variations to the point-wise conformal transformations, we prove that the critical points of functional are metrics of constant Ricci-directional curvature. Finally, the gradient flow of Akbar-Zadeh curvature functional in this special direction is introduced as a good candidate for evolving to the metric with constant second type scalar curvature and we compare this new flow with introducing Ricci flow in Finsler geometry.

Keywords: Gradient flow; Curvature functional; Berger-Ebin theorem.

Subclass[2000]: 53B40, 58B20, 58E11

1 Introduction

Nonlinear heat flows were first appeared in Riemannian geometry in 1964, when Eells and Sampson have introduced the harmonic map heat flow as the gradient flow of the energy functional $E(u) = \int_M |\nabla u|^2 dV$ [1]. They have used this flow to deform given maps $u : M \rightarrow N$ between two manifolds into the extremal maps which are critical points of the energy functional $E(u)$ in the sense of the calculus of variation.

One of the fundamental problems in differential geometry is to find canonical metrics on Riemannian manifolds, that is, the metrics which are highly symmetrical, like those of constant curvature in some senses. It was first Hamilton, who used the idea of evolving an object to such an ideal state by a nonlinear heat flow and invented the Ricci flow in 1981, cf. [2].

He has proved that a Riemannian metric of strictly positive Ricci curvature on a compact 3-manifold can be deformed into a metric of positive constant curvature. Similar results for a compact n -manifold have been obtained by G. Huysen [3], C. Margerin [4] and S. Nishikawa [5].

The stationary metrics under the Ricci flow are Ricci flat metrics which are also the critical points of the Einstein-Hilbert functional $\mathcal{E}(g) = \int_M R dV$ but the Ricci flow is not exactly the gradient flow of this functional. It is just the first part of the Einstein-Hilbert functional's gradient flow, $\partial_t g_{ij} = -R_{ij} + (R/n)g_{ij}$. If this functional is restricted to the class of conformal metrics, then it has a strictly parabolic gradient flow called the Yamabe flow. Hamilton proved that there is not any functional such that its gradient flow is exactly the Ricci flow. Perelman recently improved the Einstein-Hilbert functional and introduced \mathcal{F} -entropy functional $\mathcal{F}(g_{ij}, f) = \int_M (|\nabla f|^2 + R)e^{-f} dV$. This functional had a system of PDEs such that one of these equations was the Ricci flow [6].

Moreover, the gradient flows have fruitful roles in global analysis on manifolds and in different branches of applied sciences such as image processing and biological problems. Finsler geometry is a natural generalization of Riemannian geometry. Therefore, it is natural to extend gradient flow on Finsler manifolds. For the first attempt in this topic, we can mention the concept of the Ricci flow introduced by Bao [7]. He introduced $\partial_t \log F = -Ric$ as a scalar form of the Ricci flow on Finsler manifolds since it guarantees that the metric is derived from Finsler structure without needing to check integrability conditions in each step of work. Recently, Ohta and Sturm studied the heat flows on Finsler manifolds in [8].

In the present work, the space of all Finsler metrics on a given manifold is studied. It is proved that this space has a Riemannian manifold structure and is represented by $\mathcal{M}_{\mathcal{F}}$ (see Theorem (3.2)). This manifold structure is important to make sure that the solutions of heat flows come from a Finsler structure. A heat flow is placed in the tangent space of manifold \mathcal{M}_F and if it has some solutions, then they are located in this manifold according to the concept of integral curves. So it guarantees the solutions of heat flows come from a Finsler metric no need to use the scalar form of flows. In the forth section, by means of the metric structure on manifold \mathcal{M}_F and the Berger-Ebin theorem, two different splits are given for the tangent space of $\mathcal{M}_{\mathcal{F}}$ (see Theorems 4.2, 4.3), which are the natural extension of the Riemannian case, see [9, 10, 11, 12]. In this way, in the fifth section, the variation of Finsler metrics is defined a bit more complete than what is mentioned by Akbar-Zadeh in [13] (see Definition 5.2). Furthermore, the Berger-Ebin decomposition of the tangent space \mathcal{M}_F helps us to calculate the variation of curvature functionals in different directions and find another heat flow $\partial_t g_{ij} = -H(u, u)g_{ij}$ which is a weakly parabolic equation (see Corollary 5.4). Furthermore, we define Finslerian functionals (see Definition 5.3) and we compute variations of Akbar-Zadeh functional in the point-wise conformal transformation direction and prove that the critical points of this functional are of constant Ricci-directional curvature (see Theorem 5.3).

2 Preliminaries

Let (M, g) be a connected, compact Finsler manifold. That is, there is a function F on the tangent bundle TM satisfying the following conditions:

- F is a smooth function on the entire slit tangent bundle TM_o .
- F is a positive homogeneous function on the second variable, y .
- The matrix (g_{ij}) , $g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is non-degenerate.

2.1 Connections on Finsler manifold

Geodesics of a Finsler structure F are characterized locally by $\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0$, where $G^i = \frac{1}{4} g^{ih} (\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j - \frac{\partial F^2}{\partial x^h})$ are called geodesic spray coefficients. Let $G_j^i = \frac{\partial G^i}{\partial y^j}$ be the coefficients of a nonlinear connection on TM . By means of this nonlinear connection, the tangent space TM_o splits into horizontal and vertical subspaces. TTM_o spanned by $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, where $\frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - G_j^i \frac{\partial}{\partial y^j}$ are called Berwald bases and their dual bases are denoted by $\{dx^i, \delta y^i\}$, where $\delta y^i := dy^i + G_j^i dx^j$. Furthermore, this nonlinear connection can be used to define a linear connection called the Berwald connection and its connection 1-forms are defined locally by $\pi_j^i = G_{jk}^i dx^k$ where $G_{jk}^i = \frac{\partial G_j^i}{\partial y^k}$. The connection 1-forms of the Cartan connection are defined by $\tilde{\nabla} \frac{\partial}{\partial x^i} = \omega_j^i \frac{\partial}{\partial x^j}$, where $\omega_j^i = \Gamma_{jk}^i dx^k + C_{jk}^i \delta y^k$ such that

$$\Gamma_{jk}^i = \frac{1}{2} g^{im} (\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^m}) - (C_{js}^i G_k^s + C_{ks}^i G_j^s - C_{kjs} G^{si}),$$

and

$$C_{jk}^i = \frac{1}{2} g^{im} (\frac{\partial g_{mj}}{\partial y^k} + \frac{\partial g_{mk}}{\partial y^j} - \frac{\partial g_{kj}}{\partial y^m}), \quad (1)$$

Hence we have $\tilde{\nabla} = \nabla + \dot{\nabla}$ where, ∇ is the horizontal coefficients of the Cartan connection and $\dot{\nabla}$ is the vertical coefficients of the Finslerian(Cartan) connection. For more details of this subsection see [14].

2.2 The curvature tensors of Finsler manifold

The hh-curvature of the Cartan and Berwald connections are denoted respectively by $R_j^i{}_{kl}$ and $H_j^i{}_{kl}$. They are related by [13],

$$R_j^i{}_{kl} = H_j^i{}_{kl} + C_{jr}^i R_o^r{}_{kl} + \nabla_l \nabla_o C_{jk}^i - \nabla_k \nabla_o C_{jl}^i + \nabla_o C_{lr}^i \nabla_o C_{jk}^r - \nabla_o C_{kr}^i \nabla_o C_{jl}^r,$$

The Ricci tensor is defined in different ways in Finsler geometry. In the present work, we consider the Akbar-Zadeh's definition of Ricci tensor given by $\tilde{H}_{ij} = 1/2 \frac{\partial^2}{\partial y^i \partial y^j} (H_{rs} y^r y^s)$, where $H_{ij} = g^{ks} H_{ikjs}$. The Ricci-directional curvature is defined by $H(u, u) = g^{ik} H_{ijkl} u^j u^l$ where, $u^i = \frac{y^i}{F}$. The scalar curvature of second type is defined by $\tilde{H} = g^{ij} \tilde{H}_{ij}$. According to the above formula, it is not important which connections are used for defining the Ricci tensor and the Ricci-directional curvature.

2.3 The Indicatrix bundle

Let $x_0 \in M$, define $S_{x_0}M = \{y \in T_{x_0}M | F(x_0, y) = 1\}$ which is called Indicatrix at the point x_0 . Put $SM := \bigcup_{x_0 \in M} S_{x_0}M$, SM is called the Indicatrix bundle of a Finsler structure F . All the geometric objects on SM are positive homogeneous of degree zero. If f is a function on SM and ∂_α 's ($1 \leq \alpha \leq n-1$) are partial derivatives along Indicatrix then its derivative is $\partial_\alpha f = v_\alpha^i \frac{\partial f}{\partial y^i}$ where $\partial_\alpha = v_\alpha^i \frac{\partial}{\partial y^i}$ and v_α^i is the transition matrix of rank $(n-1)$. So the coefficients of the induced metric on SM are $g_{\alpha\gamma} = v_\alpha^i v_\gamma^j g_{ij}$ and since the vertical Liouville vector field L is normal to the Indicatrix with respect to this metric, we have

$$(g_{ij})_{1 \leq i, j \leq n} = \begin{bmatrix} (g_{\alpha\gamma}) & 0 \\ 0 & 1 \end{bmatrix}.$$

The dual bases of ∂_α is denoted by β^α and is defined by $\beta^\alpha = \nabla u = \omega_n^\alpha$, cf. [15].

The Indicatrix bundle SM is always orientable and the compactness of M provided that SM is compact, too. These two properties of SM permit us to define integral on Finsler manifolds and a global inner product on SM . The volume element of the Indicatrix bundle is denoted by $(2n-1)$ -form η , cf. [13],

$$\eta := \frac{(-1)^N}{(n-1)} \phi, \quad \phi = \omega \wedge (d\omega)^{(n-1)}, \quad N = \frac{n(n-1)}{2}. \quad (2)$$

where ω is the Hilbert form. On the tensor spaces on SM , The canonical (point-wise) scalar product is denoted by $\langle . | . \rangle$ and the global scalar product on their sections is denoted by $(. | .) = \int_{SM} \langle . | . \rangle \eta$. The codifferential operator on the space of differentiable 1-forms is defined on SM by, cf. [13],

$$\delta a = -(\nabla^j a_j - a_j \nabla_0 C^j), \quad (3)$$

where, a is a horizontal 1-form on SM . And

$$\delta b = -F(\dot{\nabla}_j b^j + b_j C^j) = -F g^{ij} \partial_j b_i. \quad (4)$$

where, b is a vertical 1-form on SM .

2.4 Curvature functional

Akbar-Zadeh defined different functionals by means of different curvature tensors in [13, 16]. The more general case among them is

$$I(g_t) = \int_{SM} \hat{H}_t \eta_t, \quad (5)$$

where $\hat{H} = \tilde{H} - c(x)H(u, u)$. The critical points of this functional are called generalized Einstein metrics. More preciesly, we have the following:

Definition 2.1. A Finslerian manifold is called a generalized Einstein manifold (GEM) if the Ricci-directional curvature is independent of the direction. That is to say

$$\tilde{H}_{ij}(x, y) = C(x)g_{ij}(x, y).$$

Through finding critical points of the functional $I(g_t)$, Akbar-Zadeh proved that

$$C(x) = nH(u, u) = \tilde{H}. \quad (6)$$

So the Ricci-directional curvature is related to the second type scalar curvature, see [13] for more details.

3 The space of Finsler metrics

The space of Riemannian metrics on a given manifold is an infinite dimensional manifold. It is easy to see this property since the Riemannian metrics space is the open and convex set of the space of all sections of S^2T^*M . Ebin used the manifold structure in [9] and gave a Riemannian structure to the manifold of Riemannian metrics on a compact manifold M . The aim of this section is to consider the geometry of the space of Finslerian metrics. Dealing with Finslerian case is not as easy as Riemannian case because of PDEs and integrability conditions for defining the Finsler metrics. The outline of the proof is to start by the generalized Lagrange metrics and restricted it to find a suitable PDE for introducing Finsler metric space. The generalized Lagrange metric is a metric structure on π^*TM or VTM and is defined as follows:

Definition 3.1. A generalized Lagrange metric, briefly a GL-metric on an n -dimensional manifold M , is a $(0, 2)$ d-type tensor field $g_{ij}(x, y)$ on TM satisfying the following

- $g_{ij}(x, y) = g_{ji}(x, y)$, i.e. it is symmetric,
- $\det g_{ij}(x, y) \neq 0$, i.e. it is regular,
- The quadratic form $g_{ij}(x, y)\xi^i\xi^j$, $\xi \in \mathbb{R}^n$ has a constant signature.

If we only consider positive signature, then $g(x, y)$ is a Euclidean product of the vector space $\pi^*|_z TM$ for each $z = (x, y) \in U \subset TM$. So π^*TM is a Riemann vector bundle over TM . A GL-metric is called a Lagrange metric, if there is a potential function $L : TM \rightarrow \mathbb{R}$ such that

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}(x, y), \quad (7)$$

are components of a positive definite matrix. A GL-metric is reducible to a Lagrange metric if and only if the Cartan tensor (16) is symmetric in all three indices. This condition is equivalent to the integrability condition of the system (7) i.e. $\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial g_{ik}}{\partial y^j}$ is satisfied. It signifies that the equation (1) is reduced to the form $C_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{2} \partial_k g_{ij}$. Furthermore, the coefficients of a Finslerian metric are zero homogeneous, so they are lying on SM . Hence a Lagrange metric is reduced to a Finsler metric if and only if the coefficients of the metric are satisfied with a system of the linear partial differential equations, $y^k \frac{\partial g_{ij}}{\partial y^k} = 0$, see [17] for more details. So the problem of introducing the space of Finsler metrics is reduced to finding the solution space of the following system:

$$\begin{cases} y^i \partial_i g_{jk} = 0; & i, j, k = 1, \dots, n \\ g(\xi, \xi) > 0; & \xi \in \Gamma(\pi^*TM_0) \end{cases} \quad (8)$$

We note that since these equations are defined in L-metrics space so the potential function is always defined by

$$L(x, y) = g_{ij}(x, y)y^i y^j.$$

for the solutions of (8). It means that the integrability condition is satisfied for these solutions. Now, the procedure is to define another system of equations which is equivalent to (8).

Definition 3.2. Let E and F be vector bundles over the manifold M . A linear differential operator of order q from E to F is a map $\phi o j^q : E \rightarrow F$ between the sets of germs of sections E and F where, $\phi : J^q(E) \rightarrow F$ is a vector bundle morphism and $J^q(E)$ is the jet bundle of E of order q .

A GL-metric is a field of cones on $S^2\pi^*T^*M$, that is

$$\begin{aligned} k : TM &\rightarrow S^2\pi^*T^*M \\ z &\rightarrow k(z) \subset E_z \end{aligned} \tag{9}$$

where $k(z) = \{g_{ij} \in S^2\pi^*T^*M | \det g_{ij} > 0\} \cup \{g_{ij} \in S^2\pi^*T^*M | \det g_{ij} < 0\}$. So the space of GL-metrics is a symmetric 2-forms bundle over TM endowed with a field of cones which is denoted by $E := [S^2\pi^*T^*M; K]$ cf. [18]. Let F be the subbundle of J^1E which is spanned at each point $z \in TM$ by $(u^{ij}, u_k^{ij}, u_\alpha^{ij}, u_L^{ij})$ where, $L = y^k \frac{\partial}{\partial y^k}$ is the vertical Liouville vector field. Suppose that $P : \Gamma(E) \rightarrow \Gamma(F)$ is a linear first order differential operator which is defined by $P(g) := \Phi o j^1(g) = y^k \partial_k g_{ij}$ c.f [19].

Definition 3.3. A morphism of vector bundles $\sigma(P) : S^q T^*M \otimes E \rightarrow F$ which is fibered over $P : E \rightarrow F$ is called the symbol of P .

The symbol of P is defined by:

$$\begin{aligned} \sigma(P) : T^*(TM) \otimes E &\rightarrow F \\ \sigma_t(P) &= P(fg), \end{aligned}$$

where $t = df$. In local coordinate, we have $P(fg) = y^k \partial_k (fg_{ij})$. So by means of the integrability condition for system (8), the kernel of this symbol is the space of conformal Finsler metrics. For any $s \geq 3$, the vector space

$$V_s := (T^*TM \otimes E) \cap (S^{s-1}T^*TM \otimes \ker(\sigma(P))),$$

is vanish. Therefore, the system (P, E, F) is of finite type. So the equation $P(g) = 0$ is equivalent to the closed system of PDEs of the form $\partial_k g_{ij} = \psi_k(ij)$ where, $\psi_k(ij)$ are a combination of the homogeneous functions of order -1 of y^i , $L(x, y)$, $\frac{\partial L}{\partial y^i}(x, y)$ and $\frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$. Hence the system of equations (8) is equivalent to the following system:

$$\begin{cases} \partial_k g_{ij} = \psi_k(ij) & ; \quad i, j, k = 1, \dots, n \\ g(\xi, \xi) > 0 & ; \quad \xi \in \Gamma(\pi^*TM_0). \end{cases} \tag{10}$$

It will thus be sufficient to prove that the system (10) has a solution, see [20]. Since this system is of finite type, i.e. the higher order derivatives can be written in lower order derivatives, the integrability condition is always true for this system.

Proposition 3.1. *The system of PDEs (10) has a solution.*

Proof. The 1-forms associated with this linear system are $d^v g_{ij} - \psi_k(ij) dy^k = 0$, so the annihilator of these 1-forms are $X_k = \partial_k + \psi_k(ij) \frac{\partial}{\partial g_{ij}}$. According to the Frobenius theorem, this system has a solution if and only if $[X_k, X_l] = 0$. This condition is equivalent:

$$\partial_k \psi_l(ij) - \partial_l \psi_k(ij) + \psi_l(mn) \frac{\partial \psi_k(ij)}{\partial g_{mn}} - \psi_k(mn) \frac{\partial \psi_l(ij)}{\partial g_{mn}} = 0. \quad (11)$$

By integrability condition, we have $\partial_k \psi_l(ij) = \partial_l \psi_k(ij)$, so this equation is reduced to

$$\psi_l(mn) \frac{\partial \psi_k(ij)}{\partial g_{mn}} - \psi_k(mn) \frac{\partial \psi_l(ij)}{\partial g_{mn}} = 0. \quad (12)$$

Represent the set of algebraic equations (12) by $F_K(z, g) = 0$, where $K = 1, \dots, n^2$. So we have maximum n^2 independent and it yields that the system of equations (10) has a solution according to Theorem (2.1) of [20]. \square

Theorem 3.2. *The space of all Finsler metrics on a compact manifold M is a Riemannian manifold.*

Proof. Let F be a solution of (12) so F is a homogeneous function of order 2 on TM . It means that the solution space of (12) is an infinite dimensional manifold. Suppose g is a solution of (10), so $g \in \Gamma(E)$ such that it is zero homogeneous and satisfies in the integrability condition i.e. $g_{ij} = 1/2 \frac{\partial^2 F}{\partial y^i \partial y^j}$. So the solution space of (10) is an infinite dimensional manifold, too. This solution space is represented by \mathcal{M}_F . For every $g \in \mathcal{M}_F$, the tangent space of this manifold is the space of all symmetric 2-forms which are positive homogeneous of degree zero and symmetric in all three indices i.e.

$$T_g \mathcal{M}_F = \{h \in S^2(\pi_s^* T^* M) | \partial_j h_{ik} = \partial_k h_{ij}\},$$

Define the global inner product on \mathcal{M}_F by

$$(a, b)_g := \int_{SM} \langle a, b \rangle \eta, \quad (13)$$

where $a, b \in T_g \mathcal{M}_F$. The local inner product is defined by $\langle a, b \rangle := g^{-1} a g^{-1} b$ and we suppose that a and b are square integrable. This inner product smoothly depends on g . Therefore, the pair $(\mathcal{M}_F, \langle \cdot, \cdot \rangle)$ is an infinite dimensional Riemannian manifold. \square

4 Different decompositions of the tangent space of \mathcal{M}_F

It is well known that $\pi^* TM$ is isomorphic to VTM . Let us consider a section $s : M \rightarrow TM$. The pullback bundle $s^* VTM$ is a vector bundle over M and for all $x \in M$ there is an isomorphism $\Pi_x : (VTM)_{s(x)} \rightarrow (s^* VTM)_x \cong (s^* \pi^* TM)_x$. We use this isomorphism frequently without notification in this work. Consider a vector field $V \in \Gamma(TM)$ and denote by η_t the 1-parameter local flow of V . Let $\tilde{\eta}$ be the natural extension of η on TM defined by

$\tilde{\eta}_t : (x^i, y^i) \rightarrow (x^i + tv^i, y^i + ty^m \frac{\partial v^i}{\partial x^m})$. Clearly, $\hat{V} := \frac{d}{dt}|_{t=0} \tilde{\eta}_t$ is the complete lift of the vector field V on TM .

Let $X = X^i \frac{\partial}{\partial x^i}$ be a section of $\pi_s^* TM$. Consider the canonical linear mapping $\varrho : T_z TM \rightarrow \pi_s^* T_x M$ which is defined by $\varrho_z(\frac{\delta}{\delta x^i}) = \frac{\partial}{\partial x^i}|_x$ and $\varrho_z(\frac{\partial}{\partial y^i}) = 0$ in local coordinates. Suppose \hat{X}, \hat{Y} and \hat{Z} are sections of TTM so by using the Lie derivative and torsion definitions and the properties of Cartan connection we obtain:

$$\begin{aligned} L_{\hat{X}}g(\varrho\hat{Y}, \varrho\hat{Z}) &= L_{\hat{X}}g(Y, Z) \\ &= g(\text{symm}(\nabla X)\hat{Y}, Z) + g(Y, \text{symm}(\dot{\nabla} X)\hat{Z}) \\ &\quad + 2g(T(X, \dot{Z}), Y) + g(T(\dot{X}, Z), Y) + g(T(\dot{X}, Y), Z), \end{aligned}$$

where $\dot{X} := \nabla_{\hat{X}} L$ and $g(\text{symm}(\nabla X)\hat{Y}, Z) := g(\nabla_{H\hat{Y}} X, Z) + g(Y, \nabla_{H\hat{Z}} X)$. It is defined similarly for vertical connection.

Now, Let \hat{X} be the complete lift of a vector field X on M . Replacing this vector field in the above Lie derivative equation, and using $y^m \frac{\partial X^i}{\partial x^m} = y^m \frac{\delta X^i}{\delta x^m}$, we get

$$\begin{aligned} \nabla_{(X^i G_i^l + y^i \frac{\partial X^l}{\partial x^i}) \frac{\partial}{\partial y^l}} \frac{\partial}{\partial x^k} &= (X^i G_i^l + y^i \frac{\partial X^l}{\partial x^i}) C_{lk}^m \frac{\partial}{\partial x^m} \\ &= (y^m \frac{\delta X^l}{\delta x^m} + y^m X^r F_{rm}^l) C_{lk}^m \frac{\partial}{\partial x^m} \\ &= y^i \nabla_i X^l C_{lk}^m \frac{\partial}{\partial x^m}. \end{aligned}$$

So in local coordinate, we deduce that

$$L_{\hat{X}}g(\varrho\hat{Y}, \varrho\hat{Z}) = \nabla_i X_j + \nabla_j X_i + 2y^m \nabla_m X^k C_{kij}. \quad (14)$$

By means of the global inner product (13), we define the adjoint of this operator.

Lemma 4.1. *Let (M, g) be a compact Finslerian manifold and h an arbitrary symmetric 2-form in $S^2 \pi_s^* T^* M$. Then the adjoint of Lie derivative of h in local coordinates is given by*

$$\delta h = -(\nabla^i h_{ik} - h_{kj} \nabla_0 C^j + \dot{C}_{kij} h^{ij} + C_{kij} \nabla_o h^{ij}), \quad (15)$$

Proof.

$$\begin{aligned} \int_{SM} \frac{1}{2} (L_{\hat{X}} g, h) \eta &= \frac{1}{2} \int_{SM} (\nabla_i X_j + \nabla_j X_i + 2y^m \nabla_m X^k C_{ijk}) h^{ij} \eta \\ &= \int_{SM} \nabla_i X_j h^{ij} \eta + \int_{SM} y^m \nabla_m X^k C_{kij} h^{ij} \eta \\ &= \int_{SM} (h_{ik} \nabla_0 C^i - \nabla^i h_{ij} - (\nabla_0 C_{ijk}) h^{ij} - C_{ijk} \nabla_0 h^{ij}) X^k \eta \\ &= - \int_{SM} (\nabla^i h_{ik} - h_{ik} \nabla_0 C^i + \dot{C}_{kij} h^{ij} + C_{kij} \nabla_o h^{ij}) X^k \eta \\ &= \int_{SM} (X, \delta h) \eta. \end{aligned}$$

□

Theorem 4.2. *The Berger-Ebin decomposition of $T_g\mathcal{M}_F \subset S^2\pi_s^*T^*M$ is $T_g\mathcal{M}_F = \{h|h = L_{\hat{X}}g\} \oplus S^T$ where $S^T := \{h|\delta_g h = 0\}$.*

Proof. Define the differential operator τ_g for every $g \in \mathcal{M}_F$ by

$$\begin{aligned}\tau_g : \Gamma(TM) &\rightarrow T_g\mathcal{M}_F \\ \tau_g(X) &:= L_{\hat{X}}g,\end{aligned}$$

where \hat{X} is the complete lift of X . The adjoint of this operator is denoted by τ_g^* and defined as follows:

$$\begin{aligned}\tau_g^* : T_g\mathcal{M}_F &\rightarrow \Gamma(TM) \\ \tau_g^*(h) &= -\sharp\delta_g h.\end{aligned}$$

For an arbitrary vertical 1-form t on SM , the symbol of τ is defined by:

$$\sigma_t(\tau) = t \otimes X_{\sharp} + X_{\sharp} \otimes t,$$

It is injective so the Berger-Ebin decomposition of $T_g\mathcal{M}_F$ is as follows:

$$T_g\mathcal{M}_F = \text{Im}\tau_g \oplus \ker\tau_g^*, \quad (16)$$

where $\text{Im}\tau_g = \{h|h = L_{\hat{X}}g\}$ and $\ker\tau_g^* = \{h|\delta_g h = 0\}$. \square

Remark 4.1. By means of decomposition (16), every h in $T_g\mathcal{M}_F$ is decomposed as $h = h_0 + L_{\hat{X}}g$. So this decomposition is unique up to the Finslerian Killing vector fields.

The point-wise conformal deformation of a Finslerian metric g is defined $\tilde{g}(x, y) = f(x)g(x, y)$ where, f is a smooth positive function on M , [21]. Since there is a one to one correspondence between the space of positive functions and space of exponential functions by $f \rightarrow e^f$, we can write $\tilde{g} = e^f g$. Let \mathcal{P} be the product group of positive functions on M that acts on \mathcal{M}_F as follows:

$$\begin{aligned}A : \mathcal{P} \times \mathcal{M}_F &\rightarrow \mathcal{M}_F \\ A(f, g) &:= fg,\end{aligned}$$

This action is free and smooth. The orbit of this action at $g \in \mathcal{M}_F$ is defined by $A_g = \{fg|f \in \mathcal{P}\}$ which is a submanifold of \mathcal{M}_F [12]. The tangent space of this submanifold at g is defined by $\mathcal{F}g = \{h = kg|k \in C^\infty(M)\}$ which is a subbundle of $S^2\pi_s^*T^*M$ at each point $g \in \mathcal{M}_F$. The orthogonal subspace of $\mathcal{F}g$ with respect to the global inner product is $S^T := \{h \in S^2\pi_s^*T^*M | \int_{SM} kgh\eta = 0\} = \{h \in S^2\pi_s^*T^*M | \text{tr}(h) = 0\}$. On the other hand, by means of the variation of volume forms [13], $\text{tr}(h) = 0$ if and only if SM has constant volume. So the orthogonal space of $\mathcal{F}g$ is the space of 2-forms which preserve volume SM through metric variations. Thus, there is a point-wise decomposition like

$$T_g\mathcal{M}_F = \mathcal{F}g \oplus S^T. \quad (17)$$

Let \mathcal{D} be the group of infinitesimal diffeomorphism on M and \mathcal{P} be a 1-parameter group of positive function on M . Put $\mathcal{C} = \mathcal{D} \times \mathcal{P}$ which is a semi-direct group with the following action:

$$(\eta_1, f_1) \cdot (\eta_2, f_2) = (\eta_1 \circ \eta_2, f_2(f_1 \circ \eta_2)),$$

This group acts on \mathcal{M}_F by function \tilde{A} as follows:

$$\begin{aligned} \tilde{A} : \mathcal{C} \times \mathcal{M}_F &\rightarrow \mathcal{M}_F \\ \tilde{A}((\eta, f), g) &= f(\tilde{\eta}^* g), \end{aligned}$$

The orbit of \tilde{A} passing through $g \in \mathcal{M}_F$ is

$$\begin{aligned} \tilde{A}_g : \mathcal{C} &\rightarrow \mathcal{M}_F \\ \tilde{A}_g(\eta, g) &= f(\tilde{\eta}^* g), \end{aligned}$$

which is a submanifold of \mathcal{M}_F [12].

Theorem 4.3. *The York decomposition of $\mathcal{B} \subset T_g \mathcal{M}_F$ is $\mathcal{B} = \mathcal{F}g \oplus S^{TT} \oplus (S^T \cap \text{Im} \tau_g)$, where \mathcal{B} is defined as the solution space of the system $\frac{\partial h_j^i}{\partial y^k} = 0$.*

Proof. Define $\tau_g := d\tilde{A}_g|_{(e,1)}$ as follows:

$$\begin{aligned} \tau_g : \Gamma(TM) \times C^\infty(M) &\rightarrow T_g \mathcal{M}_F \\ \tau_g(X, k) &= L_{\hat{X}} g + kg, \end{aligned}$$

The adjoint of τ_g is denoted by τ_g^* and defined by:

$$\begin{aligned} \tau_g^* : T_g \mathcal{M}_F &\rightarrow \Gamma(TM) \times C^\infty(M) \\ h &\rightarrow (\sharp \text{div} h, \text{tr}(h)), \end{aligned}$$

The condition $\frac{\partial h_j^i}{\partial y^k} = 0$ leads to the function $\text{tr}(h)$ is just function of x . So τ^* is well-defined. The kernel of this map is $S^{TT} = \{h \in T_g \mathcal{M}_F | \text{div} h = 0, \text{tr}(h) = 0\}$, and since the symbol of the map τ_g i.e. $\sigma_t(\tau_g)(X, f) = fg + t \otimes X_\sharp + X_\sharp \otimes t$ where, t is an arbitrary vertical 1-form on SM is injective so the Berger-Ebin decomposition is

$$T_g \mathcal{M}_F = S^{TT} \oplus \text{Im} \tau_g,$$

By corresponding this decomposition with point-wise decomposition (17), we get

$$T_g \mathcal{M}_F = \mathcal{F}g \oplus S^{TT} \oplus (S^T \cap \text{Im} \tau_g). \quad (18)$$

□

Example 1. The subset \mathcal{B} of $T_g \mathcal{M}_F$ is nonempty. Let $F(x, y)$ be a Finsler structure which does not reduce to the Riemannian case. Suppose $\tilde{g} = e^{f(t,x)} g(x, y)$ is an arbitrary curve in \mathcal{M}_F . So $h_i^j = \delta_i^j e^{f_0(x)}$ only depends on the variable x .

The last term of equation (18) shows that every 2-form $h = L_{\hat{X}}g + fg$ preserves volume of SM that is $tr(h) = 0$. So we must have $f = -(2/n)div(\hat{X})$, that is h is in the form $h = L_{\hat{X}}g - (2/n)div(\hat{X})g$. Let $g \in \mathcal{M}_{\mathcal{F}}$ and C_g be the isotropic group of action \tilde{A} , i.e.

$$\{(\eta, f) \in \mathcal{C} | f\tilde{\eta}^*g = g\},$$

It is clear that C_g is isomorphic to the conformal deformation group, i.e.

$$\{\eta \in \mathcal{D} | \tilde{\eta}^*g = fg, \text{ for some } f \in \mathcal{P}\},$$

The Lie algebra of this group is defined by

$$K_g = \{(X, k) \in \Gamma(TM) \times C^\infty(M) | L_{\hat{X}}g + kg = 0\},$$

So it is diffeomorphic with infinitesimal conformal variation

$$\{X \in \Gamma(TM) | L_{\hat{X}}g = \frac{2}{n}div(\hat{X})g\}.$$

According to the above discussion, research works in [13] and [22] are restricted to the isotropic group of the Finsler metrics.

5 Curvature functional on $\mathcal{M}_{\mathcal{F}}$

If H is an inner product space with a smooth functional $E : H \rightarrow \mathbb{R}$, the gradient vector field $\nabla E : H \rightarrow H$ is given at each point $u \in H$ by the unique vector $\nabla E(u) \in H$ such that for all $u \in H$,

$$(\nabla E(u), V) = dE(u)V$$

Definition 5.1. The gradient flow equation of a functional E is defined as follows:

$$\begin{aligned} \frac{d}{dt}\varphi_u(t) &= -\nabla E(\varphi_u(t)) \\ \varphi_u(0) &= u_0 \end{aligned}$$

where, $\varphi : I \times H \rightarrow H$ is a curve in H .

Definition 5.2. A variation of a Finslerian metric g_o is a 1-parameter family of metrics $\{g_t\}_{t \in I}$, where $g_t = g_o + th$, $g_o \in \mathcal{M}_F$ and $h \in T_g\mathcal{M}_F$.

According to the above definition, the variation of a Finslerian metric is a curve on the manifold \mathcal{M}_F such that its tangent vector field is $h := \partial_t g_t$. When a Finslerian metric is deformed, then the geometric structures, like nonlinear coefficients, curvature tensors, volume forms and Indicatrix will be changed as well. Variations of these objects are calculated in [13],

$$\eta' = (g^{ij} - \frac{n}{2}u^i u^j)h_{ij}\eta. \quad (19)$$

$$V(t)' = \frac{1}{2} \int_{SM} tr(h)\eta = \frac{n}{2} \int_{SM} t(u, u)\eta \quad (20)$$

$$G_k^i = \frac{1}{2}(\nabla_k h_o^i + \nabla_o h_k^i - \nabla^i h_{ok}) - 2C_{ks}^i G'^s. \quad (21)$$

$$R_{jkl}^i = \nabla_k \Lambda_{jl}^i - \nabla_l \Lambda_{jk}^i + P_{jlr}^i \Lambda_{ok}^r - P_{jkr}^i \Lambda_{ol}^r + C_{jr}^i R_{okl}^r, \quad (22)$$

where

$$\Lambda^i_{jk} = \Gamma'^i_{jk} + C^i_{jr} \Gamma'^r_{ok}.$$

and

$$\begin{aligned} \Gamma'^i_{jk} &= \frac{1}{2} g^{im} (\nabla_k h_{mj} + \nabla_j h_{mk} - \nabla_m h_{jk}) \\ &\quad - (C^i_{js} G'^s_k + C^i_{ks} G'^s_j - C_{kjs} G'^s_m g^{im}). \end{aligned}$$

$$\hat{H}'_{jk} = \tilde{H}_{jk} - \lambda H(u, u) u_j u_k - (n\tau - \phi) u_j u_k \quad (23)$$

where

$$\begin{aligned} \tau &:= (\nabla^i \nabla_0 T_i - \nabla_0 T_i \nabla_0 T_i) + g^{ij} \partial_j (\nabla_0 \nabla_0 T_i) \\ \phi &:= \frac{1}{2} [\nabla_i \gamma^i - \gamma_i \nabla_0 T^i - F^2 g^{ij} \partial_i (\psi_j / F)] \end{aligned}$$

and

$$\gamma_i := 2\lambda \nabla_0 T_i - \nabla_i \lambda - T_i \nabla_0 \lambda$$

Definition 5.3. A real valued function E on \mathcal{M}_F is called Finslerian functional if it satisfies the condition $E((d\varphi)^*g) = E(g)$ for every diffeomorphism φ on M .

Example 2. The curvature functional (5) is a Finslerian functional. Let φ be a diffeomorphism on M and g be a Finslerian metric on manifold M so $(d\varphi)^*g \in \mathcal{M}_F$. It is easily seen that φ is an isometry between two Finslerian manifolds (M, g) and $(M, (d\varphi)^*g)$. Hence $\hat{H}_{(d\varphi)^*g_0} = \hat{H}_{g_0}$, $\eta_{(d\varphi)^*g_0} = \eta_{g_0}$ and $\tilde{S}M = SM$ and consequently $I((d\varphi)^*g) = I(g)$. So the functional (5) only depends on Finslerian geometric data, and can be viewed as a function on the quotient space $\mathcal{M}_F/\mathcal{D}$, where \mathcal{D} denotes the diffeomorphism group of M .

Lemma 5.1. *The variation of the volume form (2) with respect to the point-wise conformal deformation at $t = 0$ is $\eta' = \frac{1}{2} \text{tr}_g(h)\eta$.*

Proof. The point-wise conformal variation of a metric g is $\tilde{g}_{ij} = e^{2f(t,x)} g_{ij}$ so $h_{ij} = \varrho(t, x) g_{ij}$, where $\varrho(t, x) = f'(t, x) e^{f(t,x)} = \frac{1}{n} \text{tr}_g(h)$. Substitute this equation in (19), at $t = 0$ we get $\eta' = \frac{1}{2} \text{tr}_g(h)\eta$. \square

Theorem 5.2. *Let (M, g) be a closed and connected Finslerian manifold with $\dim M \geq 3$. A metric g_0 at the critical point $(t = 0, g_0 = g(0))$ of the functional $I(g_t)$ is a Ricci-directional flat metric.*

Proof. Derivative of the functional $I(g_t)$ in an arbitrary direction leads to

$$A_{jk} h^{jk} = (\tilde{H}_{jk} - \lambda H(u, u) u_j u_k - (n\tau - \phi) u_j u_k - \hat{H}(g_{jk} - \frac{n}{2} u_j u_k)) h^{jk} = 0. \quad (24)$$

Since g_0 gives the extremum of $I(g_t)$ and by means of (6), \hat{H} is just a function of variable x at $t = 0$. So equation (24) reduces to

$$A_{jk} h^{jk} = (\tilde{H}_{jk} - \lambda H(u, u) u_j u_k - (n\tau - \phi) u_j u_k - \frac{1}{2} \hat{H} g_{jk}) h^{jk} = 0. \quad (25)$$

Contracting both sides of A_{jk} by u^k and u^j , we have:

$$\tilde{H}(u, u) - \lambda H(u, u) - (n\tau - \phi) - \frac{\hat{H}}{2} = 0. \quad (26)$$

By contraction of A_{jk} by g^{jk} , we obtain:

$$\tilde{H} - \lambda H(u, u) - (n\tau - \phi) - \frac{n}{2}\hat{H} = 0, \quad (27)$$

By subtracting (26) and (27), we get:

$$\frac{n-1}{2}\hat{H} = -\tilde{H} + \tilde{H}(u, u), \quad (28)$$

and

$$\lambda H(u, u) + (n\tau - \phi) = \frac{n}{n-1}\tilde{H}(u, u) - \frac{1}{n-1}\tilde{H}. \quad (29)$$

Replacing two last equations in A_{jk} and contracting by u^j and u^k we have:

$$H(u, u) = 0.$$

Hence proof is complete. \square

Remark 5.1. If in the final step, we contract the equation by g^{jk} instead of u^j and u^k then we obtain $\tilde{H} = 0$.

This functional is not invariant under the rescaling. For eliminating this problem, we use a normal factor $\psi = \psi(t)$, and put $\tilde{g} = \psi(t)g(t)$ such that $\int_{SM} \tilde{\eta} = 1$. Therefore, $\eta = \psi^{\frac{-n}{2}}\tilde{\eta}$ and by replacing it in the volume formula, we have $\psi = (V(t))^{\frac{-2}{n}}$. Next, we rewrite the functional $I(g_t)$ with respect to this normalized factor

$$\begin{aligned} \tilde{I}(g) &= I(\tilde{g}_t) = \int_{SM} (H(\tilde{g}) - \lambda H(u, u)(\tilde{g}))\tilde{\eta}, \\ &= \int_{SM} \psi^{-1}(H(g) - \lambda H(u, u)(g))\psi^{\frac{n}{2}}\eta, \\ &= \psi^{\frac{n-2}{2}}I(g), \\ &= (V(t))^{\frac{2-n}{n}}I(g). \end{aligned}$$

Theorem 5.3. *Let M be a closed and connected Finslerian manifold with $\dim M \geq 3$. A metric g_0 is a critical point for $\tilde{I}(g_t)$ under all point-wise conformal variations at $t = 0$ if and only if the Finslerian manifold is of constant Ricci-directional curvature.*

Proof. Derivative of both sides of equation $\tilde{I}(g_t) = (V(t))^{\frac{2-n}{n}}I(g)$ and calculate it at $t = 0$:

$$\begin{aligned} \tilde{I}'(g_t)|_{t=0} &= \frac{2-n}{n}V(t)'|_{t=0}(V(0))^{\frac{2-n}{n}-1}I(g_0) + v(0)^{\frac{2-n}{n}}I'(g_t)|_{t=0}, \\ &= V(0)^{\frac{2-n}{n}}\left\{\frac{2-n}{2n}\frac{I(g_0)}{V(0)}\int_{SM} \text{tr}(h)\eta + \int_{SM} A_{ij}h^{ij}\eta\right\}|_{t=0}. \end{aligned}$$

Put $Ave := \frac{I(g_0)}{V(0)}$ which is a constant value. Restrict to the point-wise conformal deformation, we get:

$$\begin{aligned} 0 &= \tilde{I}'(g_t)|_{t=0} \\ &= V(0)^{\frac{2-n}{n}} \int_{SM} \left(\frac{2-n}{n} Ave + A_{ij} g^{ij} \right) \frac{tr_g(h)}{n} \eta, \end{aligned} \quad (30)$$

Since h is an arbitrary 2-form in $\mathcal{F}g$, we have:

$$0 = \frac{2-n}{2} Ave + A_{ij} g^{ij} = \frac{2-n}{2} Ave - \tilde{H} + \lambda H(u, u) + (n\tau - \phi) + \frac{n}{2} \hat{H}, \quad (31)$$

Substituting (28) and (29) into (31) and using (6) we obtain:

$$H(u, u) = -\frac{(n-2)}{4n} Ave.$$

□

Remark 5.2. In the set of stationary points of the curvature functional $I(g_t)$, based on the constant Indicatrix volume, we have (6). Hence \tilde{H} is constant, as well.

Corollary 5.4. *The unnormalized gradient flow of $\tilde{I}(g_t)$ with respect to the subspace $\mathcal{F}g$ of $T_g\mathcal{M}_F$ is*

$$\frac{\partial}{\partial t} g_{ij}(t, z) = -H_t(u, u) g_{ij}(t, z). \quad (32)$$

and it is a strictly parabolic equation.

Proof. Derivative of the functional $I(g_t)$ is given by

$$I'(g_t)|_{t=0} = \int_{SM} H(u, u) tr_g(h) \eta = \int_{SM} H(u, u) g_{jk} h^{jk} \eta = 0$$

so its Euler-Lagrange equation is given by $H(u, u) g_{ij} = 0$. It follows that its associated gradient flow is $\frac{\partial}{\partial t} g_{ij} = -H_t(u, u) g_{ij}(t)$. The linearization of this equation is

$$\begin{aligned} D[H(u, u) g_{ij}] : C^\infty(S^2(T^*TM)) &\rightarrow C^\infty(S^2(T^*TM)), \\ D[H(u, u) g_{ij}] \left(\frac{\partial g_{ij}}{\partial t} \right) &= D[H(u, u)](h_{ij}) = \frac{\partial}{\partial t} \tilde{H}(u, u). \end{aligned}$$

So we have

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{H}(u, u) &= \frac{\partial}{\partial t} \left(\tilde{H}_{ij} \frac{y^i}{\tilde{F}} \frac{y^j}{\tilde{F}} \right), \\ &= \tilde{F}^{-2} (\nabla_s \nabla_0 h_0^s - \frac{1}{2} \nabla_s \nabla^s h_{00} - \nabla_0 \nabla_0 h_s^s + \text{lower order terms}) \tilde{g}_{ij}. \end{aligned}$$

The total symbol of the Ricci directional curvature $H(u, u)$ is

$$\sigma[H(u, u) g_{ij}](\xi)(h_{ij}) = \tilde{F}^{-2} (h_0^i \xi_i \xi_s y^s - \frac{1}{2} \xi_i \xi^i h_{00} - y^s \xi_s y^l \xi_l h_i^i) g_{ij} + \text{lower order terms}.$$

So the principal symbol of the tensor $H(u, u)g_{ij}$ is

$$\hat{\sigma}[H(u, u)g_{ij}](\xi)(h_{ij}) = \tilde{F}^{-2}(h_0^i \xi_i \xi_s y^s - \frac{1}{2} \xi_i \xi^i h_{00} - y^s \xi_s y^l \xi_l h_i^i) \tilde{g}_{ij}.$$

Put $\xi_1 = 1$ and $\xi_j = 0$ for all $j \neq 1$. To evaluate the principal symbol of this equation, we take an orthonormal frame (e_i) at $x \in M$ such that $u^n = \frac{y^n}{F} = 1$ and $u^\alpha = 0$ for all $\alpha \neq n$, it is clear that

$$\begin{aligned} \hat{\sigma}[H(u, u)g_{ij}](x, y)(\xi, h_{ij}) &= (h_i^i \xi_i \xi_s u^s u^l - \frac{1}{2} h(u, u) - u^s \xi_s u^l \xi_l \text{tr}(h)) \tilde{g}_{ij}, \\ &= -\frac{1}{2} h(u, u) \tilde{g}_{ij} = -\partial_t \log \tilde{F} \tilde{g}_{ij}. \end{aligned}$$

Hence it is a strictly parabolic equation. \square

The normalized gradient flow of functional $I(g_t)$ with restricted to the point-wise conformal deformation is $\frac{\partial}{\partial t} g_{ij}(z, t) = -(H_t(u, u) - c(t))g_{ij}(t, z)$, where $c(t)$ is a constant value at each t and is defined by $c(t) = \frac{\int_{SM} \hat{H}_t \eta_t}{\int_{SM} \eta_t}$.

Remark 5.3. According to the above discussion, we can define (32) by means of \tilde{H} , i.e.

$$\partial_t g_{ij}(t, z) = -\tilde{H}_t g_{ij}(t, z).$$

According to the Akbar-Zadeh's calculations, the Euler-Lagrange equation of functional (5) for an arbitrary direction is $-\tilde{H}_{ij} + c(x)g_{ij} = 0$. So its associated gradient flow is

$$\frac{\partial g_{ij}}{\partial t}(t, z) = -\tilde{H}_{ij}(t, z) + H_t(u, u)g_{ij}(t, z) = -\nabla I(g_t) \quad (33)$$

Consider the linearization of this equation. Since $\tilde{H}_{ij}' = \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (H'_{kr} y^k y^r)$ and $H'_{ks} y^k y^s = 2\nabla_r G'^r - \nabla_0 G'^r_r + 2\nabla_0 T_r G'^r$, cf. [13] and use (21), we deduce that the first term of (33) is of order 4 in term of h . Note that similar to the Riemannian case, (33) has not any solution since the second term of (33) is a backward equation. Bao considered the first term of (33) as the Ricci flow on Finsler manifolds, i.e. $\partial_t g_{ij} = -\tilde{H}_{ij}(t)$. Through this work, we derive the second term of (33), i.e. $\partial_t g_{ij} = -H_t(u, u)g_{ij}(t)$ as a gradient flow in the special direction of variations. Both of these flows have scalar form $\partial_t \log F_t = -H_t(u, u)$. So we prefer to use the tensor forms of flows for our later studying and use manifold \mathcal{M}_F to guarantees their solutions come from Finsler structure.

References

- [1] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, *American Journal of Mathematics*, Vol. 86, No. 1 (Jan., 1964), 109-160.
- [2] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, *J. Differential Geom.* 17 (1982), no. 2, 255-306.

- [3] G. Huysken, *Ricci deformation of the metric on a Riemannian manifold*, *J. Differential Geom.* 21 (1985), 47-62.
- [4] C. Margerin, *Point-wise pinched manifolds are space forms*, *Proc. Sympos. Pure Math.* 44 (1986), 307-328.
- [5] S. Nishikawa, *Deformation of Riemannian metrics and manifolds with bounded curvature ratios*, *Proc. Sympos. Pure Math.*, Vol. 44 (1986), 343-352.
- [6] R. Muller, *Differential Harnack Inequalities and the Ricci Flow*, *EMS Series of Lectures in Mathematics*, 2006. arxiv: 11041647.v1, preprint. Accessed 2012.
- [7] D. Bao, *On two curvature-driven problems in Riemann-Finsler geometry*, *Adv. stud. pure Math.* 48 (2007), 19-71.
- [8] Sh. Ohta and K. T. Sturm, *Heat flow on Finsler manifolds*, *arXive.org.*, 2012.
- [9] D. Ebin, *The manifold of Riemannian metrics*, *Proc. Symp. Pure Math.*, 15, 1140 (1970).
- [10] M. Berger and D. Ebin, *Some decompositions of the space of symmetric tensors on a Riemannian manifold*, *J. Differ. Geom.*, 3, No. 3, 379-392 (1969).
- [11] E. Fischer and E. Marsden, *The manifold of conformally equivalent metrics*, *Can. J. Math.*, Vol; XXIX, no 1, 193-209 (1977).
- [12] N. K. Smolentsev, *Spaces of Riemannian metrics*, *Journal of Mathematical Sciences*, Vol; 142, No. 5, 2007.
- [13] H. Akbar-Zadeh, *Generalized Einstein manifolds*, *Journal of Geometry and Physics*, Vol; 17, 342-380 (1995).
- [14] D. Bao, S.S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, *Graduate Texts in Mathematics*, vol; 200, Springer , 2000.
- [15] H. Akbar-Zadeh, *Sur les espaces de Finsler la courbures sectionnelles constantes*, *Acad. Roy. Belg. Bull. Cl. Sci.* 74 (1988), 281-322.
- [16] H. Akbar-Zadeh, *Initiation to Global Finslerian Geometry*, *North-Holland Mathematical Library*, vol; 68, 2006.
- [17] I. Bucataru and R. Miron, *Finsler-Lagrange geometry, Applications to dynamical systems*, *CEEX ET 3174/2005-2007 and CEEX M III 12595/2007*.
- [18] A. Bllga and R. D. Ene, *Field of cones on a polysymplectic manifold*, *Analele stiintifice Ale university AT "AL.I. CUZA" DIN IAS I (S.N.) MATEMATICA*, Tomul LIII, 2007, f.1.
- [19] J. Grifon and Z. Muzsnay, *Variational Principles for Second-order Differential Equations*, *World Scientific Publishing Co. Pte. Ltd., British Library Cataloguing-in-Publicatlon Data*, (1923).

- [20] *M. Dunajski, Overdetermined PDEs, University of Cambridge, 2008.*
- [21] *M. S. Knebelman, Conformal geometry of generalized metric spaces, Proc. nat. Acad. Sci. USA 15, 376-379(1929).*
- [22] *P. Joharinad and B. Bidabad, Conformal vector fields on Finsler spaces, Differential Geometry and its Applications, Volume 31, Issue 1, February 2013, 33-40*
- [23] *D. Bao and C. Robles, Ricci and flag curvatures in Finsler geometry, MSRI pub, Vol; 50, (2004).*
- [24] *A. L. Besse, Einstein Manifolds, Springer-Verlag, Berlin (1987).*
- [25] *B. Chow and D. Knopf, The Ricci Flow, An Introduction, Mathematical Surveys and Monographs 110, Amer. Math. Soc., Providence, RI, 2004.*